Destruction of states in quantum mechanics

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Abstract
A description of the destruction of states on the grounds of quantum mechanics
rather than quantum field theory is proposed. Several kinds of maps called
supertraces are defined and used to describe the destruction procedure. The
introduced algorithm can be treated as a supplement to the von Neumann–
Lüders measurement. The discussed formalism may be helpful in a description
of EPR-type experiments and in quantum information theory.

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1. Introduction

In this paper, we propose a solution to the following problem: how to describe the destruction
of a particle on the level of quantum mechanics with finite degrees of freedom. This question
arises when Einstein–Podolsky–Rosen-type experiments [1] (see also, e.g., [2]) or the tests of
quantum mechanical state reduction (see, e.g., [3]) are studied. In this type of experiment,
two particles are produced in an entangled state and sent to two measurement devices at
a distance where correlated quantities are measured at the same time. Prediction of the
correlation between the data does not cause any problems in such an ideal experiment, but if
both measurements are not really performed at the same time we have to take into account
the fact that a particle is irreversibly absorbed by a detector during measurement. This has
nothing in common with the annihilation of a particle in quantum field theory; therefore, to
avoid any confusion we shall use the word 'destruction' to describe this kind of processes.

    Evidently, if we take into account the destruction we have to consider open quantum
mechanical systems. We make the idealization relying on the assumption that the destruction
process is instantaneous, therefore its description should not involve any dynamics. For this
reason the methods of quantum field theory are not appropriate for our purpose since QFT
can be applied to open systems only if the dynamics is given, e.g. by coupling the fields to
external classical sources. Moreover, in QFT formalism one has to use an infinite direct sum
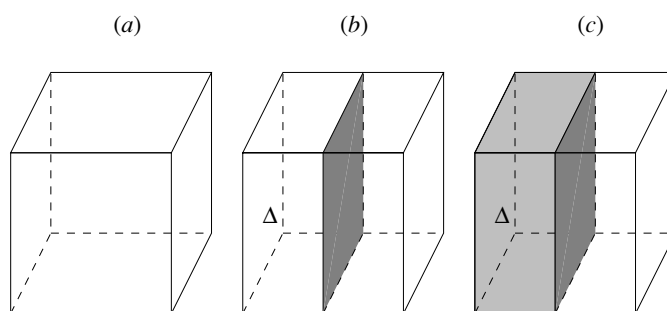
Figure 1. Destruction of a particle in a part of a box: (*a*) there is a particle in the box, (*b*) the box is divided by a barrier, (*c*) destruction in the region Δ—there is no particle in the grey part of the box.

of tensor product Hilbert spaces (asymptotic Fock space) while we would like to describe quantum systems with finite degrees of freedom.

Destruction of a particle in a detector usually occurs when some quantum numbers (e.g. spin, position or momentum) of the particle belong to a specified subset of spectrum of the corresponding observable. Therefore, we must have a quantum system and a detector which checks if the particle quantum numbers are inside a given subset of spectrum. If the answer is 'yes', the particle is destroyed.

In this paper, we introduce a mathematical framework which allows us to define the destruction process based on the principles of quantum mechanics. The physical examples of destruction, including spatial localization of the particles as well as application of the destruction to the calculation of quantum correlations, will be given in forthcoming papers.

This paper is organized as follows. In section 2 we consider destruction of a one-particle state, first intuitively, then formally. In the next section we discuss the space of states necessary for the description of destruction of two-particle states. In section 4 we introduce supertraces and study their basic properties. Sections 5 and 6 deal with the destruction of two-particle systems of distinguishable and identical particles, respectively. We illustrate each of these cases by examples.

2. Destruction of one particle

We begin with the discussion of a toy model in which the destruction of a single particle takes place in a given region of space. In the framework of this model we formulate a description of the process of destruction of a one-particle state taking physical intuition as a guiding principle. And then we consider the general case, not necessarily related to the localization of the particle.

Thus, let us consider a box containing one particle (see figure 1(*a*)) in the state given by the density matrix ρ. Now we divide the box into two parts (e.g. by a non-penetrating barrier—figure 1(*b*)). We destroy the particle if it is inside the region Δ of the box (figure 1(*c*)).

First, let us discuss the situation when we check if the particle is inside Δ. It means that we first perform a measurement with the selection of the observable Π_Δ, where Π_Δ is the projector onto the subspace of the states localized in Δ. The measurement of Π_Δ gives either 1 if the particle is inside the region Δ, or 0 if it is outside Δ. The particle is destroyed if the measurement of Π_Δ gives 1, i.e. its state is replaced by the vacuum state. Thus, in this case, the destruction procedure is done in two *immediate* steps:

(i) the initial density matrix ρ is reduced to

$$\rho' = \begin{cases} \dfrac{\Pi_\Delta \rho \Pi_\Delta}{\mathrm{Tr}(\rho \Pi_\Delta)} & \text{if the particle is inside } \Delta \\[2ex] \dfrac{\Pi_\Delta^\perp \rho \Pi_\Delta^\perp}{\mathrm{Tr}(\rho \Pi_\Delta^\perp)} & \text{if the particle is outside } \Delta \end{cases} \tag{1}$$

where $\Pi_\Delta^\perp = I - \Pi_\Delta$ (I denotes the identity operator);

(ii) if $\rho' = \Pi_\Delta \rho \Pi_\Delta / \mathrm{Tr}(\rho \Pi_\Delta)$, then it is mapped onto vacuum density matrix ρ_{vac}, otherwise it is left unchanged, so

$$\rho'' = \begin{cases} \rho_{\mathrm{vac}} & \text{if the particle is inside } \Delta \\[2ex] \dfrac{\Pi_\Delta^\perp \rho \Pi_\Delta^\perp}{\mathrm{Tr}(\rho \Pi_\Delta^\perp)} & \text{if the particle is outside } \Delta. \end{cases} \tag{2}$$

But what happens if we place the barrier, but do not check if the particle is inside Δ? This situation corresponds to a measurement with no selection of the observable Π_Δ. The particle is either inside Δ with the probability $\mathrm{Tr}(\rho \Pi_\Delta)$ or outside Δ with the probability $\mathrm{Tr}(\rho \Pi_\Delta^\perp)$, thus

(i) first, the density matrix ρ is reduced to

$$\rho' = \Pi_\Delta \rho \Pi_\Delta + \Pi_\Delta^\perp \rho \Pi_\Delta^\perp \tag{3}$$

(ii) then, after the destruction we get either the vacuum with the probability $\mathrm{Tr}(\rho \Pi_\Delta)$ or the one-particle state with the probability $\mathrm{Tr}(\rho \Pi_\Delta^\perp)$, so

$$\rho'' = \Pi_\Delta^\perp \rho \Pi_\Delta^\perp + \mathrm{Tr}(\rho \Pi_\Delta) \rho_{\mathrm{vac}}. \tag{4}$$

It is easy to see that in both cases the map $\rho \mapsto \rho''$ is linear on the combinations $\mu \rho_1 + (1 - \mu) \rho_2$, where $\mu \in [0, 1]$ and ρ_1, ρ_2 are the density matrices, i.e. in the convex set of density matrices.

Now, let us rewrite the above procedure in a slightly more abstract and general context, not necessarily related to the localization of a particle. Let $\mathcal{H}$ be the Hilbert space of states for a particle. The one-particle states (density matrices) form a convex subset of the endomorphism space of $\mathcal{H}$ (i.e. $\rho \in \mathrm{End}(\mathcal{H})$). In order to describe the system if destruction occurs we must introduce the vacuum state $|0\rangle$ and one-dimensional vacuum space spanned by $|0\rangle$, i.e. $\mathcal{H}^0 \equiv \mathrm{span}\{|0\rangle\}$. The vacuum vector $|0\rangle$ is orthogonal to any vector from $\mathcal{H}$ and every observable acts trivially on it. Therefore, the Hilbert space of the system under consideration is a direct sum $\mathcal{H} \oplus \mathcal{H}^0$, and the states are mixtures of the elements from $\mathrm{End}(\mathcal{H})$ and $\mathrm{End}(\mathcal{H}^0)$. Furthermore, let $\hat{\Lambda}$ be an arbitrary observable with the spectrum Λ and Ω be a subset of the spectrum. Denote the subspace spanned by all the eigenvectors corresponding to the eigenvalues from the subset Ω by $\mathcal{H}_\Omega$ and the projector onto this subspace by Π_Ω. If the particle state is an element of $\mathrm{End}(\mathcal{H}_\Omega)$ then the particle is destroyed, otherwise it is not.

Therefore, let us find a linear map from $\mathrm{End}(\mathcal{H})$ to $\mathrm{End}(\mathcal{H}^0)$ which leaves the trace invariant. It is enough to restrict ourselves to the endomorphisms of the form $|\chi\rangle\langle\phi|$, where $|\chi\rangle, |\phi\rangle \in \mathcal{H}$. This map must act on these endomorphisms in the following way:

$$\mathrm{End}(\mathcal{H}) \ni |\chi\rangle\langle\phi| \mapsto c|0\rangle\langle0| \in \mathrm{End}(\mathcal{H}^0). \tag{5}$$

Because $\mathrm{Tr}(|\chi\rangle\langle\phi|) = \langle\phi|\chi\rangle$ and $\mathrm{Tr}(c|0\rangle\langle0|) = c$, it follows that $c = \langle\phi|\chi\rangle$. Therefore, this leads to the following definition.

Definition 1. *The supertrace* $\widehat{\mathrm{Tr}}$ *is a linear map* $\widehat{\mathrm{Tr}} : \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H}^0)$ *such that its action on the endomorphism of the form* $|\chi\rangle\langle\phi| \in \mathrm{End}(\mathcal{H})$ *is given by the following formula*

$$\widehat{\mathrm{Tr}}(|\chi\rangle\langle\phi|) = \langle\phi|\chi\rangle|0\rangle\langle0|. \tag{6}$$

We call $\widehat{\mathrm{Tr}}$ a supertrace[1] because it is a superoperator, i.e. it is the operator in the endomorphism space (see, e.g., [4]).

It is easy to check that if the set of vectors $\{|a\rangle\}$ is an orthonormal basis[2] in $\mathcal{H}$ and $\hat{L} = \sum_{aa'} L_{aa'} |a\rangle\langle a'| \in \mathrm{End}(\mathcal{H})$ is a linear operator, then

$$\widehat{\mathrm{Tr}}(\hat{L}) = \sum_{aa'} L_{aa'} \delta_{aa'} |0\rangle\langle0| = \mathrm{Tr}(\hat{L})|0\rangle\langle0| \tag{7}$$

($\delta_{aa'}$ denotes the Kronecker delta).

Applying the $\widehat{\mathrm{Tr}}$ operation to the Ω-projected part of ρ (i.e. $\Pi_\Omega \rho \Pi_\Omega$) we can formalize the procedure which gave us the density matrix ρ'' by the following definitions.

Definition 2. *A destruction with selection in the set* Ω *of one-particle state* $\rho \in \mathrm{End}(\mathcal{H})$ *is defined by the map*[3] $D_\Omega^s : \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H}) \oplus \mathrm{End}(\mathcal{H}^0)$, *such that*

$$D_\Omega^s(\rho) = \begin{cases} \dfrac{\widehat{\mathrm{Tr}}(\Pi_\Omega \rho \Pi_\Omega)}{\mathrm{Tr}(\rho \Pi_\Omega)} & \text{if the measurement of } \Pi_\Omega \text{ gives } 1 \\[3mm] \dfrac{\Pi_\Omega^\perp \rho \Pi_\Omega^\perp}{\mathrm{Tr}(\rho \Pi_\Omega^\perp)} & \text{if the measurement of } \Pi_\Omega \text{ gives } 0. \end{cases} \tag{8}$$

Definition 3. *The destruction with no selection in the set* Ω *of a one-particle state* $\rho \in \mathrm{End}(\mathcal{H})$ *is defined by the map* $D_\Omega : \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H}) \oplus \mathrm{End}(\mathcal{H}^0)$, *such that*

$$D_\Omega(\rho) = \Pi_\Omega^\perp \rho \Pi_\Omega^\perp + \widehat{\mathrm{Tr}}(\Pi_\Omega \rho \Pi_\Omega). \tag{9}$$

Note that D_Ω^s and D_Ω are superoperators. In quantum information theory superoperators similar to D_Ω are considered as choice superoperators describing the coherent information transfer between subsets of the entire system [5].

It is easy to check that applying the destruction maps D_Ω^s and D_Ω to the density matrix ρ describing a state of a particle in a box (see above), we get the density matrices ρ'' from (2) and (4), respectively, when $\hat{\Lambda}$ is the position operator, $\Omega = \Delta$ and $\rho_{\mathrm{vac}} = |0\rangle\langle0|$.

We have to show that the endomorphisms $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$, which we get after the destruction, are density matrices. In other words, we have to prove that D_Ω^s and D_Ω are the Kraus maps [6]. This is guaranteed by the following proposition.

Proposition 1. *The superoperators* D_Ω^s *and* D_Ω *from the definitions* 2 *and* 3, *respectively, are Kraus maps.*

Proof. Indeed, $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$ are Hermitian because $\Pi_\Omega^\dagger = \Pi_\Omega$, and $\mathrm{Tr}(\rho \Pi_\Omega)$ and $\mathrm{Tr}(\rho \Pi_\Omega^\perp)$ are real. Next, $\mathrm{Tr}(\widehat{\mathrm{Tr}}(\Pi_\Omega \rho \Pi_\Omega)) = \mathrm{Tr}(\rho \Pi_\Omega)$, so $\mathrm{Tr}(D_\Omega^s(\rho)) = 1$. Because $\Pi_\Omega^\perp = I - \Pi_\Omega$, we have

$$\mathrm{Tr}(D_\Omega(\rho)) = \mathrm{Tr}(\rho \Pi_\Omega^\perp) + \mathrm{Tr}(\rho \Pi_\Omega) = \mathrm{Tr}(\rho) = 1. \tag{10}$$

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[1] We point out to avoid confusion that this supertrace has nothing in common with the supertrace Str used in supersymmetry.

[2] If we consider continuous bases, we must replace sums and Kronecker deltas by integrals and Dirac deltas, respectively.

[3] We shall frequently use the more general and shorter term 'map' instead of 'superoperator' if it does not lead to misunderstandings.

The proof that $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$ are non-negative is obvious. $\Pi_\Omega^\perp \rho \Pi_\Omega^\perp$ is non-negative because it is an orthogonal projection of a non-negative ρ. $\widehat{\mathrm{Tr}}(\Pi_\Omega \rho \Pi_\Omega) = \mathrm{Tr}(\rho \Pi_\Omega)|0\rangle\langle 0|$ is non-negative because $\mathrm{Tr}(\rho \Pi_\Omega) \geqslant 0$. Thus $D_\Omega^s(\rho)$ is non-negative. $D_\Omega(\rho)$ is also non-negative, because it is the sum of two non-negative terms, which act in orthogonal subspaces. So the maps D_Ω^s and D_Ω are the Kraus maps. $\qquad\square$

We now illustrate the destruction procedure in the case when the observable $\hat{\Lambda}$ is not the position operator by the following example.

Example 1. Consider a spin-$\frac{1}{2}$ particle. We assume that the destruction *with no selection* takes place if the z-component of the spin is $\frac{1}{2}$. In this case $\hat{\Lambda} = \hat{S}_3$ and its spectrum is $\Lambda = \{-\frac{1}{2}, \frac{1}{2}\}$ and $\Omega = \{\frac{1}{2}\}$. The one-particle Hilbert space is $\mathcal{H} = \mathrm{span}\{|\uparrow\rangle, |\downarrow\rangle\}$, the subspace $\mathcal{H}_\Omega = \mathrm{span}\{|\uparrow\rangle\}$ and the corresponding projection operator is $\Pi_\Omega = |\uparrow\rangle\langle\uparrow|$, so $\Pi_\Omega^\perp = |\downarrow\rangle\langle\downarrow|$. The most general density matrix in this case is

$$\rho = w|\uparrow\rangle\langle\uparrow| + c|\uparrow\rangle\langle\downarrow| + c^*|\downarrow\rangle\langle\uparrow| + (1-w)|\downarrow\rangle\langle\downarrow|$$

where $w \in [0, 1]$, $c \in \mathbb{C}$ and $|c|^2 \leqslant w(1-w)$. After the destruction we get the new state

$$D_\Omega(\rho) = w|0\rangle\langle 0| + (1-w)|\downarrow\rangle\langle\downarrow|.$$

So we get the vacuum state with the probability w and the particle with $S_3 = -\frac{1}{2}$ with the probability $1-w$.

In this case, it is easy to find the von Neumann entropy of the state before and after the destruction. The eigenvalues of ρ are $\rho_\pm = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2} - w\right)^2 + |c|^2}$, so the von Neumann entropy before destruction is

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = -\rho_+ \ln \rho_+ - \rho_- \ln \rho_-.$$

Because $\partial S(\rho)/\partial|c|^2 \leqslant 0$ for $0 \leqslant |c|^2 \leqslant w(1-w)$, then for a given value of w the entropy is maximal for the state with $c = 0$ and for these states the entropy is equal to $S(\rho) = -w \ln w - (1-w) \ln(1-w)$. When $|c|^2 = w(1-w)$ the states are pure and their entropy is $S(\rho) = 0$.

The eigenvalues of $D_\Omega(\rho)$ are w and $1-w$, so the von Neumann entropy after the destruction is

$$S(D_\Omega(\rho)) = -\mathrm{Tr}(D_\Omega(\rho) \ln(D_\Omega(\rho))) = -w \ln w - (1-w) \ln(1-w)$$

and $S(D_\Omega(\rho)) \geqslant S(\rho)$, as it was expected from the theorem that the measurements with no selection increase entropy (see [7]).

Note that the destruction *with selection* gives in this case

$$D_\Omega^s(\rho) = \begin{cases} |0\rangle\langle 0| & \text{if the measurement of } \Pi_\Omega \text{ gives 1} \\ |\downarrow\rangle\langle\downarrow| & \text{if the measurement of } \Pi_\Omega \text{ gives 0.} \end{cases}$$

Thus $S\left(D_\Omega^s(\rho)\right) = 0$ and we have

$$S\left(D_\Omega^s(\rho)\right) \leqslant S(\rho)$$

i.e. the destruction with selection can decrease entropy.

3. Destruction in two-particle system—the space of states

Now we discuss the space of states necessary for the description of destruction of two-particle states of particles 'a' and 'b'. Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be the Hilbert spaces for the particles 'a' and 'b', respectively. The two-particle Hilbert space is the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$. The state of the system is then described by the density matrix ρ, which is an endomorphism of the space $\mathcal{H}_a \otimes \mathcal{H}_b$, i.e. $\rho \in \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$. If one introduces in $\mathcal{H}_a$ an orthonormal basis $\{|a\rangle\}$ and similarly in $\mathcal{H}_b$ an orthonormal basis $\{|b\rangle\}$, then one can write the density matrix ρ in the form

$$\rho = \sum_{aa'bb'} \rho_{aba'b'}(|a\rangle \otimes |b\rangle)(\langle a'| \otimes \langle b'|) = \sum_{aa'bb'} \rho_{aba'b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|. \quad (11)$$

In the case of identical particles the two-particle Hilbert space is, of course, the projection onto the symmetric or antisymmetric part of $\mathcal{H}_a \otimes \mathcal{H}_b$, thus we must additionally require the appropriate behaviour of the coefficients $\rho_{aba'b'}$ under the exchange of indices, i.e.

$$\rho_{aba'b'} = \rho_{baa'b'} = \rho_{abb'a'} = \rho_{bab'a'} \qquad \text{(symmetric case)} \qquad (12a)$$

$$\rho_{aba'b'} = -\rho_{baa'b'} = -\rho_{abb'a'} = \rho_{bab'a'} \qquad \text{(antisymmetric case)}. \qquad (12b)$$

But such a description of the composite quantum system is not enough if we consider the measurement by the apparatus (mentioned in the previous sections) which can destroy the state. The reason is that the density matrix (11) can describe only the two-particle states of the system, while after such a measurement we could have either a one-particle state which evolves in time or a vacuum state.

This issue can be easily solved as in the case of one particle (see section 2), i.e. by introducing the one-dimensional vacuum space $\mathcal{H}^0 \equiv \mathrm{span}\{|0\rangle\}$, and taking the direct sums $\mathcal{H}_a \oplus \mathcal{H}^0$ and $\mathcal{H}_b \oplus \mathcal{H}^0$ instead of $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. The corresponding tensor product space can be decomposed in the obvious way

$$(\mathcal{H}_a \oplus \mathcal{H}^0) \otimes (\mathcal{H}_b \oplus \mathcal{H}^0) = (\mathcal{H}_a \otimes \mathcal{H}_b) \oplus ((\mathcal{H}_a \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}_b)) \oplus (\mathcal{H}^0 \otimes \mathcal{H}^0). \quad (13)$$

The first term on the right-hand side of (13), i.e. $\mathcal{H}_a \otimes \mathcal{H}_b$, describes two-particle states; the second and third terms, i.e. $(\mathcal{H}_a \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}_b)$, represent one-particle states; while the last term, $\mathcal{H}^0 \otimes \mathcal{H}^0$, is the zero-particle state. In the case of distinguishable particles we can take the terms $\mathcal{H}_a \otimes \mathcal{H}^0$ or $\mathcal{H}^0 \otimes \mathcal{H}_b$ as the Hilbert space of the system after the destruction of the particle 'b' or 'a', respectively. For identical particles we have to consider the one-particle Hilbert space as a subspace of the sum $(\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H})$, where $\mathcal{H}_a = \mathcal{H}_b \equiv \mathcal{H}$, because we do not know if the particle 'a' or 'b' was destroyed.

The bases in the endomorphism spaces of the mentioned two-, one- and zero-particle Hilbert spaces are

$$(|a\rangle \otimes |b\rangle)(\langle a'| \otimes \langle b'|) = |a\rangle\langle a'| \otimes |b\rangle\langle b'| \qquad (\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)) \qquad (14a)$$

$$(|a\rangle \otimes |0\rangle)(\langle a'| \otimes \langle 0|) = |a\rangle\langle a'| \otimes |0\rangle\langle 0| \qquad (\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}^0)) \qquad (14b)$$

$$(|0\rangle \otimes |b\rangle)(\langle 0| \otimes \langle b'|) = |0\rangle\langle 0| \otimes |b\rangle\langle b'| \qquad (\mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b)) \qquad (14c)$$

$$(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|) = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \qquad (\mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)). \qquad (14d)$$

In the case of identical particles $\mathcal{H}_a = \mathcal{H}_b = \mathcal{H}$ and we consider *the same basis* in $\mathcal{H}_a$ and $\mathcal{H}_b$, i.e. $\{|a\rangle\} = \{|b\rangle\}$. The basis maps (14a)–(14d) should then be supplemented by the basis endomorphisms

$$(|a\rangle \otimes |0\rangle)(\langle 0| \otimes \langle b'|) = |a\rangle\langle 0| \otimes |0\rangle\langle b'| \qquad (15a)$$

$$(|0\rangle \otimes |b\rangle)(\langle a'| \otimes \langle 0|) = |0\rangle\langle a'| \otimes |b\rangle\langle 0| \qquad (15b)$$

which intertwine vectors from $\mathcal{H} \otimes \mathcal{H}^0$ to $\mathcal{H}^0 \otimes \mathcal{H}$ and vice versa.

We point out that $\dim((\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H})) = 2 \dim(\mathcal{H} \otimes \mathcal{H}^0)$, so for identical particles we must choose an irreducible subspace of $(\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H})$ which corresponds to the space of one-particle states.

4. Supertraces

The partial traces $\mathrm{Tr}_a \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}_b)$ and $\mathrm{Tr}_b \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}_a)$ are widely used in various contexts (see, e.g., [8]), but they cannot be used for the description of the destruction. Thus, our purpose is, in an analogy to definition 1, to introduce maps that preserve the trace and map $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ to $\mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$, $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}^0)$ or $\mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b)$.

Let us start with the map $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$. Of course, we have

$$\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \ni |\chi\rangle\langle\phi| \otimes |\psi\rangle\langle\xi| \mapsto c|0\rangle\langle0| \otimes |0\rangle\langle0| \in \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0). \tag{16}$$

The condition that the trace must be preserved leads to $c = \mathrm{Tr}(|\chi\rangle\langle\phi| \otimes |\psi\rangle\langle\xi|) = \langle\phi|\chi\rangle\langle\xi|\psi\rangle\}$, so we can define the following linear map[4]:

Definition 4. *The tensor product supertrace* $\widehat{\mathrm{Tr}} \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$ *is a linear map such that*

$$\widehat{\mathrm{Tr}}(|\chi\rangle\langle\phi| \otimes |\psi\rangle\langle\xi|) = \langle\phi|\chi\rangle\langle\xi|\psi\rangle(|0\rangle\langle0| \otimes |0\rangle\langle0|) \tag{17}$$

for any $|\chi\rangle, |\phi\rangle \in \mathcal{H}_a$ *and* $|\psi\rangle, |\xi\rangle \in \mathcal{H}_b$. *Because of linearity, we can extend this map on the whole space* $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$.

Next, we need maps which transform the two-particle state into one-particle state. They are given by the following definition.

Definition 5. *The linear maps*

left partial supertrace $\widehat{\mathrm{Tr}}_L \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b)$

right partial supertrace $\widehat{\mathrm{Tr}}_R \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}^0)$

inner partial supertrace $\widehat{\mathrm{Tr}}_I \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}((\mathcal{H}_a \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}_b))$

external partial supertrace $\widehat{\mathrm{Tr}}_E \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}((\mathcal{H}_a \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}_b))$

act on the endomorphisms of the form $|\chi\rangle\langle\phi| \otimes |\psi\rangle\langle\xi| \in \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ *in the following way*

$$\widehat{\mathrm{Tr}}_L(|\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|) = \langle\chi|\psi\rangle(|0\rangle\langle0| \otimes |\phi\rangle\langle\xi|) \tag{18a}$$

$$\widehat{\mathrm{Tr}}_R(|\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|) = \langle\xi|\phi\rangle(|\psi\rangle\langle\chi| \otimes |0\rangle\langle0|) \tag{18b}$$

$$\widehat{\mathrm{Tr}}_I(|\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|) = \langle\chi|\phi\rangle(|\psi\rangle\langle0| \otimes |0\rangle\langle\xi|) \tag{18c}$$

$$\widehat{\mathrm{Tr}}_E(|\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|) = \langle\xi|\psi\rangle(|0\rangle\langle\chi| \otimes |\phi\rangle\langle0|). \tag{18d}$$

Because these superoperators are linear we can extend their action to the whole space $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ *since every element of* $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ *can be written as the linear combination of the endomorphisms of the form* $|\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|$.

We can see from (18c) and (18d) that the internal and external partial supertraces $\widehat{\mathrm{Tr}}_I$ and $\widehat{\mathrm{Tr}}_E$ are non-trivial only for identical particles, i.e. for symmetric or antisymmetric part of $\mathrm{End}((\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}))$ (note that in this case $\mathcal{H}_a = \mathcal{H}_b \equiv \mathcal{H}$), because in the other case $\langle\chi|\phi\rangle$ and $\langle\xi|\psi\rangle$ must vanish for any $|\psi\rangle, |\chi\rangle \in \mathcal{H}_a$ and $|\phi\rangle, |\xi\rangle \in \mathcal{H}_b$.

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[4] We use the same symbol $\widehat{\mathrm{Tr}}$ for the map $\widehat{\mathrm{Tr}} \colon \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H}^0)$ and for the map $\widehat{\mathrm{Tr}} \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$, because the second map is the generalization of the first one in the tensor product space case.

If we specify orthonormal bases $\{|a\rangle\}$ and $\{|b\rangle\}$ in the spaces $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively, then

$$\widehat{\mathrm{Tr}}_L(|a\rangle\langle a'| \otimes |b\rangle\langle b'|) = \delta_{a'a}|0\rangle\langle 0| \otimes |b\rangle\langle b'| \tag{19a}$$

$$\widehat{\mathrm{Tr}}_L(a\rangle\langle a'| \otimes |b\rangle\langle b'|) = \delta_{b'b}|a\rangle\langle a'| \otimes |0\rangle\langle 0| \tag{19b}$$

$$\widehat{\mathrm{Tr}}_I(|a\rangle\langle a'| \otimes |b\rangle\langle b'|) = \delta_{a'b}|a\rangle\langle 0| \otimes |0\rangle\langle b'| \tag{19c}$$

$$\widehat{\mathrm{Tr}}_E(|a\rangle\langle a'| \otimes |b\rangle\langle b'|) = \delta_{b'a}|0\rangle\langle a'| \otimes |b\rangle\langle 0|. \tag{19d}$$

Remark 1. Let us note that the tensor product supertrace $\widehat{\mathrm{Tr}}$ from definition 4 can be constructed as the following composition of partial supertraces

$$\widehat{\mathrm{Tr}} = \widehat{\mathrm{Tr}}_L \circ \widehat{\mathrm{Tr}}_R = \widehat{\mathrm{Tr}}_R \circ \widehat{\mathrm{Tr}}_L$$
$$\widehat{\mathrm{Tr}} = \widehat{\mathrm{Tr}}_I \circ \widehat{\mathrm{Tr}}_E = \widehat{\mathrm{Tr}}_E \circ \widehat{\mathrm{Tr}}_I.$$

Remark 2. Definition 5 can be easily generalized to the case of states of more than two particles. In such a case it is better to denote the partial supertraces by $\widehat{\mathrm{Tr}}_{ij}$, where we make the scalar product from the ith vector (ket) and jth co-vector (bra) and replace them by $|0\rangle$ and $\langle 0|$, respectively. In such a notation we have $\widehat{\mathrm{Tr}}_R \equiv \widehat{\mathrm{Tr}}_{22}$, $\widehat{\mathrm{Tr}}_L \equiv \widehat{\mathrm{Tr}}_{11}$, $\widehat{\mathrm{Tr}}_I \equiv \widehat{\mathrm{Tr}}_{21}$, $\widehat{\mathrm{Tr}}_E \equiv \widehat{\mathrm{Tr}}_{12}$. The partial supertraces which put more than one pair of $|0\rangle$ and $\langle 0|$ can be easily obtained by taking an appropriate composition of the partial supertraces $\widehat{\mathrm{Tr}}_{ij}$.

Lemma 1. *If* $\sigma \in \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ *is non-negative then* $\widehat{\mathrm{Tr}}_L(\sigma)$ *and* $\widehat{\mathrm{Tr}}_R(\sigma)$ *are non-negative.*

Proof. Let us show that $\widehat{\mathrm{Tr}}_L(\sigma)$ is non-negative for a non-negative σ. Because $\widehat{\mathrm{Tr}}_L(\sigma) \in \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b)$, we must show that $(\langle 0| \otimes \langle\phi|)\widehat{\mathrm{Tr}}_L(\sigma)(|0\rangle \otimes |\phi\rangle) \geqslant 0$ for any $|\phi\rangle \in \mathcal{H}_b$. Without loss of generality we can assume that $|\phi\rangle$ is normalized, i.e. $\langle\phi|\phi\rangle = 1$. $\mathcal{H}_b$ can be decomposed into the linear covering of $|\phi\rangle$ and the subspace $\mathcal{H}_b^\perp$ of vectors orthogonal to $|\phi\rangle$. If the set $\{|\tilde{b}\rangle\}$ is an orthonormal basis in $\mathcal{H}_b^\perp$, then the vector $|\phi\rangle$ and vectors from $\{|\tilde{b}\rangle\}$ make an orthonormal basis in $\mathcal{H}_b$. Using σ written in the basis $\{|a\rangle\}$ in $\mathcal{H}_a$ and the above basis in $\mathcal{H}_b$ and with the help of (19a) we get

$$(\langle 0| \otimes \langle\phi|)\widehat{\mathrm{Tr}}_L(\sigma)(|0\rangle \otimes |\phi\rangle) = \sum_a \sigma_{a\phi a\phi} \tag{20}$$

where $\sigma_{a\phi a\phi} = (\langle a| \otimes \langle\phi|)\sigma(|a\rangle \otimes |\phi\rangle) \geqslant 0$ which follows from the assumption that σ is non-negative. Thus, indeed, non-negativeness of σ implies non-negativeness of $\widehat{\mathrm{Tr}}_L(\sigma)$. The proof for $\widehat{\mathrm{Tr}}_R(\sigma)$ is analogous. $\qquad \square$

Note that the analogous proof of non-negativeness for the usual partial traces can be found, for example, in [8].

5. Destruction in a system of two distinguishable particles

Now we consider the destruction of a two-particle system of distinguishable particles. Let a density matrix of the form (11) describe a system of two distinguishable particles 'a' and 'b'. The apparatus mentioned in section 1 destroys the particles if the outcomes of the measurements of observables $\hat{\Lambda}_a$ and $\hat{\Lambda}_b$ lie in the subsets Ω_a and Ω_b of spectra Λ_a of $\hat{\Lambda}_a$ and Λ_b of $\hat{\Lambda}_b$, respectively. Let Π_{Ω_a} be the projector onto the subspace of $\mathcal{H}_a$ associated with Ω_a and Π_{Ω_b} be the projector onto the subspace of $\mathcal{H}_b$ associated with Ω_b. Now we perform a simultaneous measurement of the observables $\Pi_{\Omega_a} \otimes I_b$ and $I_a \otimes \Pi_{\Omega_b}$ (I_a and I_b denote the identity operators in $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively). Thus just after the measurement we have the following four possible outcomes:

(i) the measurement of $\Pi_{\Omega_a} \otimes I_b$ and $I_a \otimes \Pi_{\Omega_b}$ both give 0—there are no particles to destroy and the final state is a two-particle state;

(ii) the measurement of $\Pi_{\Omega_a} \otimes I_b$ gives 0 and the measurement of $I_a \otimes \Pi_{\Omega_b}$ gives 1—the particle 'b' is to destroy and the final state is a one-particle state of the particle 'a';

(iii) the measurement of $\Pi_{\Omega_a} \otimes I_b$ gives 1 and the measurement of $I_a \otimes \Pi_{\Omega_b}$ gives 0—the particle 'a' is to destroy and the final state is a one-particle state of the particle 'b';

(iv) the measurement of $\Pi_{\Omega_a} \otimes I_b$ and $I_a \otimes \Pi_{\Omega_b}$ both give 1—the particles 'a' and 'b' are to destroy and the final state is the vacuum state.

One can easily verify the operators $\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}$, $\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}$, $\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}$ and $\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}$, where $\Pi_{\Omega_a}^{\perp} \equiv I_a - \Pi_{\Omega_a}$ and $\Pi_{\Omega_b}^{\perp} \equiv I_b - \Pi_{\Omega_b}$ are the projectors on mutually orthogonal subspaces associated with the cases (i)–(iv), respectively. The probabilities for each of these four situations are $\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]$, $\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right]$, $\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]$ and $\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right]$, respectively.

Now, in an analogy to the definitions 2 and 3, to destroy Ω_a- and Ω_b-projected parts of the density matrix ρ we apply appropriately the $\widehat{\mathrm{Tr}}_L$ ($\widehat{\mathrm{Tr}}_R$) to the Ω_a- (Ω_b-) projected part of ρ as well as $\widehat{\mathrm{Tr}}$ to the Ω_a- and Ω_b-projected part, and we arrive at the following definitions.

Definition 6. *The destruction with selection in the set Ω of two-particle state $\rho \in$ $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ of distinguishable particles is defined by the map $D_\Omega^s \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to$ $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \oplus \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}^0) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$ of the form*

$$
D_\Omega^s(\rho) = \begin{cases}
\dfrac{\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right)}{\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]} & \text{for outcome (i)} \\[2ex]
\dfrac{\widehat{\mathrm{Tr}}_R\left[\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right]}{\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right]} & \text{for outcome (ii)} \\[2ex]
\dfrac{\widehat{\mathrm{Tr}}_L\left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]}{\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]} & \text{for outcome (iii)} \\[2ex]
\dfrac{\widehat{\mathrm{Tr}}\left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right]}{\mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right]} & \text{for outcome (iv).}
\end{cases}
\tag{21}
$$

Definition 7. *The destruction with no selection in the set Ω of two-particle state $\rho \in$ $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b)$ of distinguishable particles is defined by the map $D_\Omega \colon \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \to$ $\mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}_b) \oplus \mathrm{End}(\mathcal{H}_a \otimes \mathcal{H}^0) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}_b) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$, such that*

$$
D_\Omega(\rho) = \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right) + \widehat{\mathrm{Tr}}_R\left[\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right]
$$
$$
+ \widehat{\mathrm{Tr}}_L\left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right] + \widehat{\mathrm{Tr}}\left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right].
\tag{22}
$$

Proposition 2. *The superoperators D_Ω^s and D_Ω from definitions 6 and 7, respectively, are Kraus maps.*

Proof. The verification that $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$ are Hermitian is trivial. Taking the density matrix ρ in the form (11) one can easily check by straightforward calculation that $\mathrm{Tr}\left(D_\Omega^s(\rho)\right) = \mathrm{Tr}(\rho) = 1$ for every outcome (i)–(iv). Now,

$$
\mathrm{Tr}(D_\Omega(\rho)) = \mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right)\right] + \mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right] + \mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]
$$
$$
+ \mathrm{Tr}\left[\rho\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right] = \mathrm{Tr}(\rho) = 1.
\tag{23}
$$

$\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}^{\perp}\right)$ is an orthogonal projection of a non-negative ρ, so it is non-negative. Similarly, the entries $\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)$ and $\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)$ are non-negative. Therefore, using lemma 1 we can see that $\widehat{\mathrm{Tr}}_R \left[\left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a}^{\perp} \otimes \Pi_{\Omega_b}\right)\right]$ and $\widehat{\mathrm{Tr}}_L \left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}^{\perp}\right)\right]$ are non-negative. $\widehat{\mathrm{Tr}} \left[\left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right) \rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right]$ can be written as $\mathrm{Tr}\left[\rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right] |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ and it is non-negative because $\mathrm{Tr}\left[\rho \left(\Pi_{\Omega_a} \otimes \Pi_{\Omega_b}\right)\right] \geqslant 0$. Thus $D_{\Omega}^s(\rho)$ is non-negative. Since all these four terms act in mutually orthogonal subspaces, $D_{\Omega}(\rho)$ is non-negative too. Therefore D_{Ω}^s and D_{Ω} are Kraus maps. $\qquad\square$

Now, we illustrate the destruction of a two-particle system of distinguishable particles by the following example.

Example 2. Consider the system of spin-1 and spin-0 particles. We assume that destruction *with no selection* takes place if the z-component of the spin of each particle is 0. We have $\hat{\Lambda}_a = \hat{S}_{a3}$ and $\hat{\Lambda}_b = \hat{S}_{b3}$. So $\Lambda_a = \{-1, 0, 1\}$, $\Lambda_b = \{0\}$ and $\Omega_a = \{0\}$, $\Omega_b = \{0\}$ (note that $\Omega_b = \Lambda_b$, so the outcomes (i) and (ii) are excluded). We can take $\mathcal{H}_a = \mathrm{span}\{|1, 1\rangle, |1, 0\rangle, |1, -1\rangle\}$ and $\mathcal{H}_b = \mathrm{span}\{|0, 0\rangle\}$, where $|j, m\rangle$ are the basis vectors. The projectors can be written as $\Pi_{\Omega_a} = |1, 0\rangle\langle 1, 0|$, $\Pi_{\Omega_a}^{\perp} = |1, 1\rangle\langle 1, 1| + |1, -1\rangle\langle 1, -1|$ and $\Pi_{\Omega_b} = |0, 0\rangle\langle 0, 0|$. The most general density matrix for such a state is

$$
\begin{aligned}
\rho =\ & w_1 |1, 1\rangle\langle 1, 1| \otimes |0, 0\rangle\langle 0, 0| + c_1 |1, 1\rangle\langle 1, 0| \otimes |0, 0\rangle\langle 0, 0| + c_2 |1, 1\rangle\langle 1, -1| \otimes |0, 0\rangle\langle 0, 0| \\
& + c_1^* |1, 0\rangle\langle 1, 1| \otimes |0, 0\rangle\langle 0, 0| + (1 - w_1 - w_2)|1, 0\rangle\langle 1, 0| \otimes |0, 0\rangle\langle 0, 0| \\
& + c_3 |1, 0\rangle\langle 1, -1| \otimes |0, 0\rangle\langle 0, 0| + c_2^* |1, -1\rangle\langle 1, 1| \otimes |0, 0\rangle\langle 0, 0| \\
& + c_3^* |1, -1\rangle\langle 1, 0| \otimes |0, 0\rangle\langle 0, 0| + w_2 |1, -1\rangle\langle 1, -1| \otimes |0, 0\rangle\langle 0, 0|
\end{aligned}
$$

where the coefficients $w_1, w_2 \in [0, 1]$, $c_1, c_2, c_3 \in \mathbb{C}$ and they are restricted by the requirement that the density matrix ρ is non-negative. After the destruction we get a new state

$$
D_{\Omega}(\rho) = w_1 |1, 1\rangle\langle 1, 1| \otimes |0\rangle\langle 0| + w_2 |1, -1\rangle\langle 1, -1| \otimes |0\rangle\langle 0| + (1 - w_1 - w_2)|0\rangle\langle 0| \otimes |0\rangle\langle 0|
$$

(recall that $|0\rangle$ denotes the *vacuum* vector), so the new state is a mixture of the spin-1 particle in the up direction (with the probability w_1), the spin-1 particle in the down direction (with the probability w_2) and the vacuum (with the probability $1 - w_1 - w_2$).

6. Destruction in a system of two identical particles

Now we consider the destruction in a state of two identical particles. In this case $\mathcal{H}_a = \mathcal{H}_b \equiv \mathcal{H}$. A system of two identical particles is described by a density matrix of the form (11) together with the symmetry conditions (12*a*) or (12*b*). As in previous cases, let Π_{Ω} be the projector onto the subspace of $\mathcal{H}$ associated with $\Omega \subset \Lambda$. Now we perform a measurement of the symmetrized observable $\Pi_{\Omega} \otimes I + I \otimes \Pi_{\Omega}$. The spectral decomposition of this observable is

$$
\Pi_{\Omega} \otimes I + I \otimes \Pi_{\Omega} = 0 \cdot \Pi_{\Omega}^{\perp} \otimes \Pi_{\Omega}^{\perp} + 1 \cdot (\Pi_{\Omega}^{\perp} \otimes \Pi_{\Omega} + \Pi_{\Omega} \otimes \Pi_{\Omega}^{\perp}) + 2 \cdot \Pi_{\Omega} \otimes \Pi_{\Omega} \qquad (24)
$$

($\Pi_{\Omega}^{\perp} = I - \Pi_{\Omega}$, as before), where

$\Pi_{\Omega}^{\perp} \otimes \Pi_{\Omega}^{\perp}$ corresponds to the situation that there is no particle with an eigenvalue of $\hat{\Lambda}$ belonging to Ω,

$\Pi_{\Omega}^{\perp} \otimes \Pi_{\Omega} + \Pi_{\Omega} \otimes \Pi_{\Omega}^{\perp}$ corresponds to the situation that there is exactly one particle with an eigenvalue of $\hat{\Lambda}$ belonging to Ω,

$\Pi_{\Omega} \otimes \Pi_{\Omega}$ corresponds to the situation that there are two particles with an eigenvalue of $\hat{\Lambda}$ belonging to Ω.

In view of (24), just after the measurement, we have only three possibilities:

(i) the measurement of $\Pi_\Omega \otimes I + I \otimes \Pi_\Omega$ gives 0—there is no particle to destroy and the final state is a two-particle state,

(ii) the measurement of $\Pi_\Omega \otimes I + I \otimes \Pi_\Omega$ gives 1—there is exactly one particle to destroy and the final state is a one-particle state,

(iii) the measurement of $\Pi_\Omega \otimes I + I \otimes \Pi_\Omega$ gives 2—there are two particles to destroy and the final state is the vacuum state.

The probabilities that one of the three cases (i)–(iii) occurs are $\mathrm{Tr}[\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)]$, $\mathrm{Tr}[\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega + \Pi_\Omega \otimes \Pi_\Omega^\perp)]$ and $\mathrm{Tr}[\rho(\Pi_\Omega \otimes \Pi_\Omega)]$, respectively.

In order to destroy the Ω-projected part of the density matrix ρ we apply the same algorithm as in the case of distinguishable particles, but now we cannot omit $\widehat{\mathrm{Tr}}_I$ and $\widehat{\mathrm{Tr}}_E$ because their action is non-trivial. Therefore, we can formulate the following definitions.

Definition 8. *The destruction with selection in the set Ω of two-particle state $\rho \in \mathrm{End}(\mathcal{H} \otimes \mathcal{H})$ of identical particles is defined by the map $D_\Omega^s: \mathrm{End}(\mathcal{H} \otimes \mathcal{H}) \to \mathrm{End}(\mathcal{H} \otimes \mathcal{H}) \oplus \mathrm{End}((\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H})) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$, such that*

$$D_\Omega^s(\rho) = \frac{(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)}{\mathrm{Tr}[\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)]} \tag{25a}$$

$$\begin{aligned} D_\Omega^s(\rho) = \{&\widehat{\mathrm{Tr}}_R[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}_L[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \\ &\pm \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \pm \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)]\} \\ &\times \{\mathrm{Tr}[\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \mathrm{Tr}[\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)]\}^{-1} \end{aligned} \tag{25b}$$

$$D_\Omega^s(\rho) = \frac{\widehat{\mathrm{Tr}}[(\Pi_\Omega \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega)]}{\mathrm{Tr}[\rho(\Pi_\Omega \otimes \Pi_\Omega)]} \tag{25c}$$

for the outcomes (i), (ii) *and* (iii) *of the measurement of $\Pi_\Omega \otimes I + I \otimes \Pi_\Omega$, respectively; where the signs $+$ and $-$ correspond to symmetric and antisymmetric cases, respectively.*

Definition 9. *The destruction with no selection in the set Ω of two-particle state $\rho \in \mathrm{End}(\mathcal{H} \otimes \mathcal{H})$ of identical particles is defined by the map $D_\Omega: \mathrm{End}(\mathcal{H} \otimes \mathcal{H}) \to \mathrm{End}(\mathcal{H} \otimes \mathcal{H}) \oplus \mathrm{End}((\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H})) \oplus \mathrm{End}(\mathcal{H}^0 \otimes \mathcal{H}^0)$, such that*

$$\begin{aligned} D_\Omega(\rho) = (\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp) + \widehat{\mathrm{Tr}}_R[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] \\ + \widehat{\mathrm{Tr}}_L[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \pm \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \\ \pm \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}[(\Pi_\Omega \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega)] \end{aligned} \tag{26}$$

where the signs $+$ and $-$ correspond to symmetric and antisymmetric cases, respectively.

In view of the discussion at the end of 3, we shall show the following lemma.

Lemma 2. *For a symmetric or antisymmetric density matrix $\rho \in \mathrm{End}(\mathcal{H} \otimes \mathcal{H})$ the state given by (25b) belongs to the irreducible one-particle subspace of $\mathrm{End}((\mathcal{H} \otimes \mathcal{H}^0) \oplus (\mathcal{H}^0 \otimes \mathcal{H}))$ (the signs $+$ and $-$ correspond to symmetric and antisymmetric cases, respectively).*

Proof. Let the sets of vectors $\{|\beta\rangle\}$ and $\{|\alpha\rangle\}$ be the orthonormal basis in $\mathcal{H}_\Omega$ and $\mathcal{H}_\Omega^\perp$, respectively. So the set $\{|\alpha\rangle\} \cup \{|\beta\rangle\}$ is a basis in $\mathcal{H}$. Let us write the density matrix ρ in the

form (11) using this basis. Moreover, we can write $\Pi_\Omega = \sum_\beta |\beta\rangle\langle\beta|$ and $\Pi_\Omega^\perp = \sum_\alpha |\alpha\rangle\langle\alpha|$. Therefore, using the symmetry conditions (12*a*) or (12*b*), we get

$$
\widehat{\mathrm{Tr}}_R[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}_L[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)]
$$
$$
\pm \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \pm \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)]
$$
$$
= \sum_{\alpha\alpha'}\left(\sum_\beta \rho_{\alpha\beta\alpha'\beta}\right)(|\alpha\rangle\otimes|0\rangle + |0\rangle\otimes|\alpha\rangle)(\langle\alpha'|\otimes\langle0| + \langle0|\otimes\langle\alpha'|) \tag{27}
$$

so, it belongs to the one-particle irreducible subspace of $\mathrm{End}((\mathcal{H}\otimes\mathcal{H}^0)\oplus(\mathcal{H}^0\otimes\mathcal{H}))$. □

Proposition 3. *The superoperators D_Ω^s and D_Ω from definitions 8 and 9, respectively, are Kraus maps.*

Proof. To prove that $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$ are Hermitian, we have only to check if the sum $\widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp\otimes\Pi_\Omega)\rho(\Pi_\Omega\otimes\Pi_\Omega^\perp)]+\widehat{\mathrm{Tr}}_E[(\Pi_\Omega\otimes\Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp\otimes\Pi_\Omega)]$ is Hermitian, since the remaining parts of (25*a*) or (26) are evidently Hermitian. First, observe that $((\Pi_\Omega^\perp\otimes\Pi_\Omega)\rho(\Pi_\Omega\otimes\Pi_\Omega^\perp))^\dagger = (\Pi_\Omega\otimes\Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp\otimes\Pi_\Omega)$. Now, it is easy to see from definition 5 that for any endomorphism $\sigma \in \mathrm{End}(\mathcal{H}\otimes\mathcal{H})$ we have $(\widehat{\mathrm{Tr}}_I(\sigma))^\dagger = \widehat{\mathrm{Tr}}_E(\sigma^\dagger)$ and vice versa. Therefore

$$
(\widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] + \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)])^\dagger
$$
$$
= \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)]. \tag{28}
$$

Thus $D_\Omega^s(\rho)$ and $D_\Omega(\rho)$ are Hermitian. In order to prove that $\mathrm{Tr}(D_\Omega^s(\rho)) = \mathrm{Tr}(D_\Omega(\rho)) = \mathrm{Tr}(\rho)$ it is enough to note that the diagonal elements of the internal and external partial supertraces vanish. This is evident from (19*c*) and (19*d*). By virtue of this fact, the rest of the proof of this point is analogous to the proof of the respective part of proposition 2. $(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega^\perp)$ and $\widehat{\mathrm{Tr}}[(\Pi_\Omega \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega)]$ are of course non-negative. The proof that the sum

$$
\widehat{\mathrm{Tr}}_R[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}_L[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)]
$$
$$
\pm \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \pm \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] \tag{29}
$$

is non-negative is the following. Let $|\phi\rangle\otimes|0\rangle+|0\rangle\otimes|\phi\rangle$ be the vector from $(\mathcal{H}\otimes\mathcal{H}^0)\oplus(\mathcal{H}^0\otimes\mathcal{H})$. The vector $|\phi\rangle \in \mathcal{H}$ can be decomposed as follows $|\phi\rangle = c|x\rangle + d|y\rangle$, where $|x\rangle \in \mathcal{H}_\Omega^\perp$, $|y\rangle \in \mathcal{H}_\Omega$, $c, d \in$ set $\mathbb{C}$ and $\langle x|x\rangle = \langle y|y\rangle = 1$. Next, we construct the basis in the subspace $\mathcal{H}_\Omega^\perp$ as in the proof of lemma 1, with the vector $|x\rangle$ basis vector. Now, using (27) we get

$$
(\langle\phi|\otimes\langle0| + \langle0|\otimes\langle\phi|)(\widehat{\mathrm{Tr}}_R[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)] + \widehat{\mathrm{Tr}}_L[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)]
$$
$$
\pm \widehat{\mathrm{Tr}}_I[(\Pi_\Omega^\perp \otimes \Pi_\Omega)\rho(\Pi_\Omega \otimes \Pi_\Omega^\perp)] \pm \widehat{\mathrm{Tr}}_E[(\Pi_\Omega \otimes \Pi_\Omega^\perp)\rho(\Pi_\Omega^\perp \otimes \Pi_\Omega)])
$$
$$
\times (|\phi\rangle\otimes|0\rangle + |0\rangle\otimes|\phi\rangle) = 4|c|^2\sum_\beta \rho_{x\beta x\beta}. \tag{30}
$$

Clearly the sum in (30) is non-negative, since ρ is non-negative. Thus $D_\Omega^s(\rho)$ is also non-negative. Since the sum (29) and the other terms in (26) act in mutually orthogonal subspaces, $D_\Omega(\rho)$ is also non-negative. Therefore D_Ω^s and D_Ω are Kraus maps. □

Now, we illustrate the destruction of two-particle system of identical particles by the following example.

Example 3. Consider the system of two identical spin-$\frac{1}{2}$ particles. We assume that the destruction *with no selection* takes place if the z-component of the spin of any particle is $\frac{1}{2}$. The observable $\hat{\Lambda}$, its spectrum Λ, the subset Ω, one-particle Hilbert space $\mathcal{H}$ as well as projectors

Π_Ω and $\Pi_{\bar\Omega}^\perp$ are the same as in example 1. The two-particle space of states is antisymmetric part of $\mathcal{H} \otimes \mathcal{H}$, i.e. span$\left\{\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)\right\}$. This space is one-dimensional, thus the state is a pure one, and its density matrix is of the form

$$\rho = \tfrac{1}{2}\left(|\uparrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\uparrow|\right).$$

After the destruction of the particles with $S_3 = \tfrac{1}{2}$, we get the new state

$$\begin{aligned}D_\Omega(\rho) &= \tfrac{1}{2}\left(|\downarrow\rangle\langle\downarrow| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle 0| \otimes |0\rangle\langle\downarrow| + |0\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle 0|\right)\\ &= \tfrac{1}{2}\left(|\downarrow\rangle \otimes |0\rangle + |0\rangle \otimes |\downarrow\rangle\right)\left(\langle\downarrow| \otimes \langle 0| + \langle 0| \otimes \langle\downarrow|\right).\end{aligned}$$

So it is really an element of one-dimensional irreducible subspace of $\mathrm{End}((\mathcal{H}\otimes\mathcal{H}^0)\oplus(\mathcal{H}^0\otimes\mathcal{H}))$.

It should be noted that in this case the destruction *with selection* gives the same result.

Because before and after the destruction we deal with the pure states the von Neumann entropies of the initial and destroyed states are both equal to zero.

7. Conclusions

We have given a mathematical formalism which allows one to describe the destruction of a particle from the two-particle state in the framework of quantum mechanics. This is done by means of the reduction procedure [9] (with selection or with no selection) associated with the immediate mapping of the part of the reduced density matrix onto the vacuum density matrix and is based on the use of supertraces. We point out that the destruction procedure can be treated as a supplement to the von Neumann–Lüders measurement procedure.

Moreover, our formalism of destructions, developed for the case of one-particle and two-particle states, can be uniquely generalized to the multi-particle states, with the help of the partial supertraces $\widehat{\mathrm{Tr}}_{ij}$ (see remark 2). Also, it can be easily extended to the generalized measurements by means of positive operator-valued measures (POVM) rather than orthogonal projections.

The formalism introduced herein should be helpful in a description of the processes when one has the system under time evolution after the destruction. This may happen in the Einstein–Podolsky–Rosen type experiments (the destruction can take place in a detector). For this reason the destruction procedure may also be helpful in quantum information theory. The study of different destruction processes as well as applications of the destruction procedure to the calculation of the EPR quantum correlations will be done in forthcoming papers.

Acknowledgments

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